INSTANTANEOUS BETHE–SALPETER EQUATION: ANALYTIC APPROACH FOR NONVANISHING MASSES OF THE BOUND-STATE CONSTITUENTS

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Abstract

The instantaneous Bethe–Salpeter equation, derived from the general Bethe–Salpeter formalism by assuming that the involved interaction kernel is instantaneous, represents the most promising framework for the description of hadrons as bound states of quarks from first quantum-field-theoretic principles, that is, quantum chromodynamics. Here, by extending a previous analysis confined to the case of bound-state constituents with vanishing masses, we demonstrate that the instantaneous Bethe–Salpeter equation for bound-state constituents with (definitely) nonvanishing masses may be converted into an eigenvalue problem for an explicitly—more precisely, algebraically—known matrix, at least, for a rather wide class of interactions between these bound-state constituents. The advantages of the explicit knowledge of this matrix representation are self-evident.

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1   Introduction

In principle, the appropriate tool for the description of bound states within relativistic quantum field theory is the Bethe–Salpeter formalism. However, attempts to solve the Bethe–Salpeter equation face several, well-known obstacles. In view of this, in practical applications usually the easier-to-handle instantaneous approximation for the involved interaction kernel is considered.

In a recent paper [1], we introduced, for the somewhat simpler example of massless bound-state constituents, a technique for converting the instantaneous Bethe–Salpeter equation into an eigenvalue problem for an explicitly given matrix. Here, the analysis of Ref. [1] is extended to the case of nonvanishing masses of the bound-state constituents.

2   Bethe–Salpeter Equation as a Matrix Equation

Our goal is to demonstrate the possibility of converting the Bethe–Salpeter equation to a matrix equation. Thus, let us accept the same simplifying assumptions as in Ref. [1]:

- The propagators in the Bethe–Salpeter equation may be approximated by free propagators with some kind of effective masses of the bound-state constituents.

- The bound-state constituents have equal masses.

Paralleling the discussion in Ref. [1], we consider fermion–antifermion bound states with spin J, parity $P = (-1)^{J+1}$ and charge-conjugation quantum number $C = (-1)^J$. The corresponding equal-time Bethe–Salpeter amplitude, or "Salpeter amplitude," χ involves two independent components, Ψ_1 and Ψ_2. For two fermions of equal masses m and internal momentum $\mathbf{k}$, in the bound state's rest frame it reads in momentum space

$$\chi(\mathbf{k}) = \left[\Psi_1(\mathbf{k}) \, \frac{m - \boldsymbol{\gamma} \cdot \mathbf{k}}{E(k)} + \Psi_2(\mathbf{k}) \, \gamma^0 \right] \gamma_5 \; ;$$

$E(k) \equiv \sqrt{k^2 + m^2}$, $k \equiv |\mathbf{k}|$, is the energy of a free particle of mass m and momentum $\mathbf{k}$. We confine ourselves to the case $J = 0$, that is, to bound states with spin-parity-charge conjugation assignment $J^{PC} = 0^{-+}$, denoted by $^1\mathrm{S}_0$ in usual spectroscopic notation.

By expanding the Salpeter amplitude into some convenient set of basis matrices in Dirac space and after factorizing off the (vector) spherical harmonics depending on the angular variables, the instantaneous Bethe–Salpeter equation for fermion–antifermion bound states can be reduced to a set of coupled equations for radial wave functions [2]. For pure time-component Lorentz-vector interactions, i.e., $\gamma^0 \otimes \gamma^0$ kernels, it reads [2, 3]

$$2 \, E(k) \, \Psi_2(k) + \int_0^\infty \frac{\mathrm{d}k' \, k'^2}{(2\pi)^2} \, V_0(k, k') \, \Psi_2(k') = M \, \Psi_1(k) \; ,$$

$$2 \, E(k) \, \Psi_1(k) \tag{1}$$

$$+ \int_0^\infty \frac{\mathrm{d}k' \, k'^2}{(2\pi)^2} \left[\frac{m}{E(k)} \, V_0(k, k') \, \frac{m}{E(k')} + \frac{k}{E(k)} \, V_1(k, k') \, \frac{k'}{E(k')} \right] \Psi_1(k') = M \, \Psi_2(k) \; ,$$

where, expressed in terms of a static interaction potential $V(r)$ in configuration space,

$$V_L(k, k') \equiv 8\pi \int_0^\infty \mathrm{d}r \, r^2 \, V(r) \, j_L(k \, r) \, j_L(k' \, r) \; , \quad L = 0, 1, 2, \dots \; .$$

Here, $j_n(z)$ $(n = 0, \pm 1, \pm 2, \dots)$ are the spherical Bessel functions of the first kind [4].

Expressing, for $M \neq 0$, from the first of Eqs. (1) the component Ψ_1 in terms of the component Ψ_2 of the Salpeter amplitude χ and inserting this into the second of Eqs. (1) yields an eigenvalue equation for Ψ_2 with the bound-state mass squared as eigenvalue:

$$
\begin{aligned}
M^2 \, \Psi_2(k) \;=\; & 4\, E^2(k)\, \Psi_2(k) + 2\, E(k) \int\limits_0^\infty \frac{\mathrm{d}k'\, k'^2}{(2\pi)^2}\, V_0(k, k')\, \Psi_2(k') \\[2mm]
& + \; 2 \int\limits_0^\infty \frac{\mathrm{d}k'\, k'^2}{(2\pi)^2} \left[\frac{m^2}{E(k)}\, V_0(k, k') + \frac{k\, k'}{E(k)}\, V_1(k, k') \right] \Psi_2(k') \\[2mm]
& + \; \int\limits_0^\infty \frac{\mathrm{d}k'\, k'^2}{(2\pi)^2} \left[\frac{m}{E(k)}\, V_0(k, k')\, \frac{m}{E(k')} + \frac{k}{E(k)}\, V_1(k, k')\, \frac{k'}{E(k')} \right] \\[2mm]
& \times \; \int\limits_0^\infty \frac{\mathrm{d}k''\, k''^2}{(2\pi)^2}\, V_0(k', k'')\, \Psi_2(k'') \; .
\end{aligned}
$$

In order to convert this eigenvalue equation to matrix form, we introduce suitable sets of basis functions for the Hilbert space $L_2(R^+)$ of (with weight function $w(x) = x^2$) square-integrable functions $f(x)$ on the positive real line R^+ (see [1]). For a given value $\ell = 0, 1, 2, \dots$ of the angular momentum of their counterparts in three dimensions, the basis functions are called $\phi_i^{(\ell)}(r)$ in configuration space and $\phi_i^{(\ell)}(p)$ in momentum space. They are the same as in Ref. [1] apart from the fact that the real variational parameter $\mu > 0$ there is replaced here by the mass m of the bound-state constituents and that, consequently, normalizability of the basis vectors demands $m > 0$; their main features are summarized in Appendix A. Expanding $\Psi_2(p)$ in terms of the radial basis functions $\phi_i^{(0)}(p)$, the instantaneous Bethe–Salpeter equation is solved by diagonalizing a matrix:

$$
\mathcal{M}_{ij} = A_{ij} + B_{ij} + C_{ij}^{(1)} + C_{ij}^{(2)} + D_{ij}^{(1)} + D_{ij}^{(2)} \; ,
$$

with the abbreviations

$$
A_{ij} \;\equiv\; 4 \int\limits_0^\infty \mathrm{d}k\, k^2\, E^2(k)\, \phi_i^{(0)}(k)\, \phi_j^{(0)}(k) \; ,
$$

$$
B_{ij} \;\equiv\; \frac{2}{(2\pi)^2} \int\limits_0^\infty \mathrm{d}k\, k^2\, E(k)\, \phi_i^{(0)}(k) \int\limits_0^\infty \mathrm{d}k'\, k'^2\, V_0(k, k')\, \phi_j^{(0)}(k') \; ,
$$

$$
C_{ij}^{(1)} \;\equiv\; \frac{2\, m^2}{(2\pi)^2} \int\limits_0^\infty \frac{\mathrm{d}k\, k^2}{E(k)}\, \phi_i^{(0)}(k) \int\limits_0^\infty \mathrm{d}k'\, k'^2\, V_0(k, k')\, \phi_j^{(0)}(k') \; ,
$$

$$
C_{ij}^{(2)} \;\equiv\; \frac{2}{(2\pi)^2} \int\limits_0^\infty \frac{\mathrm{d}k\, k^3}{E(k)}\, \phi_i^{(0)}(k) \int\limits_0^\infty \mathrm{d}k'\, k'^3\, V_1(k, k')\, \phi_j^{(0)}(k') \; ,
$$

$$
D_{ij}^{(1)} \;\equiv\; \frac{m^2}{(2\pi)^4} \int\limits_0^\infty \frac{\mathrm{d}k\, k^2}{E(k)}\, \phi_i^{(0)}(k) \int\limits_0^\infty \frac{\mathrm{d}k'\, k'^2}{E(k')}\, V_0(k, k') \int\limits_0^\infty \mathrm{d}k''\, k''^2\, V_0(k', k'')\, \phi_j^{(0)}(k'') \; ,
$$

$$
D_{ij}^{(2)} \;\equiv\; \frac{1}{(2\pi)^4} \int\limits_0^\infty \frac{\mathrm{d}k\, k^3}{E(k)}\, \phi_i^{(0)}(k) \int\limits_0^\infty \frac{\mathrm{d}k'\, k'^3}{E(k')}\, V_1(k, k') \int\limits_0^\infty \mathrm{d}k''\, k''^2\, V_0(k', k'')\, \phi_j^{(0)}(k'') \; .
$$

The solution of this eigenvalue problem (of course, only for a finite matrix size d) proceeds, step by step, exactly along the lines presented in much more detail in Ref. [1]. First of all, we introduce the matrix elements of the square E^2 of the kinetic energy:

$$K_{ij}(m) \equiv \int_0^\infty \mathrm{d}k \, k^2 \, E^2(k) \, \phi_i^{(0)}(k) \, \phi_j^{(0)}(k) \ .$$

Furthermore, in order to evaluate the terms $B_{ij}, C_{ij}^{(1)}, \ldots, D_{ij}^{(2)}$ analytically—which requires repeated applications of the Fourier–Bessel transformations (6)—, we have to expand several expressions involved in these integrals in terms of the appropriate set of momentum-space basis functions $\phi_i^{(\ell)}(k)$, $\ell = 0, 1$ (cf. Eqs. (28), (31), (32) of Ref. [1]):

$$E(k) \, \phi_i^{(0)}(k) \;=\; \sum_{j=0}^{N} b_{ji}(m) \, \phi_j^{(0)}(k) \ ,$$

$$\frac{k}{E(k)} \, \phi_i^{(0)}(k) \;=\; \sum_{j=0}^{N} c_{ji} \, \phi_j^{(1)}(k) \ ,$$

$$k \, \phi_i^{(0)}(k) \;=\; \sum_{j=0}^{N} d_{ji}(m) \, \phi_j^{(1)}(k) \ ,$$

$$\frac{1}{E(k)} \, \phi_i^{(0)}(k) \;=\; \sum_{j=0}^{N} e_{ji}(m) \, \phi_j^{(0)}(k) \ .$$

As consequence of the orthonormality of the momentum-space basis functions $\phi_i^{(\ell)}(p)$, the expansion coefficients $b_{ij}(m)$, c_{ij}, $d_{ij}(m)$, and $e_{ij}(m)$ may be expressed in the form (cf. Eqs. (29), (33), (34) of Ref. [1])

$$b_{ij}(m) \;=\; \int_0^\infty \mathrm{d}k \, k^2 \, E(k) \, \phi_i^{(0)}(k) \, \phi_j^{(0)}(k) \ ,$$

$$c_{ij} \;=\; \int_0^\infty \frac{\mathrm{d}k \, k^3}{E(k)} \, \phi_i^{*(1)}(k) \, \phi_j^{(0)}(k) \ ,$$

$$d_{ij}(m) \;=\; \int_0^\infty \mathrm{d}k \, k^3 \, \phi_i^{*(1)}(k) \, \phi_j^{(0)}(k) \ ,$$

$$e_{ij}(m) \;=\; \int_0^\infty \frac{\mathrm{d}k \, k^2}{E(k)} \, \phi_i^{(0)}(k) \, \phi_j^{(0)}(k) \ .$$

These expansion coefficients are, of course, not independent but satisfy several (only in the limit $N \to \infty$, exact) relations of the kind

$$\sum_{r=0}^{N} b_{ri}(m) \, b_{rj}(m) = \sum_{r=0}^{N} d_{ri}^*(m) \, d_{rj}(m) + m^2 \, \delta_{ij} = K_{ij}(m) \ ,$$

$$\sum_{r=0}^{N} b_{ri}(m) \, e_{rj}(m) = \delta_{ij} \ ,$$

$$\sum_{r=0}^{N} d_{ir}(m) \, e_{rj}(m) = c_{ij} \ .$$

Employing these relations in order to investigate systematically the errors induced by the truncations of the expansion series in $\mathcal{M}_{ij}$, one finds that, for instance, for $d = 15$ (i.e., 15×15 matrices) and for $N = 49$ (i.e., a truncation to the first 50 basis vectors), all the above relations are satisfied with relative errors less than 3%.

Finally, we'll need the expansions of the expressions $V(r) \phi_i^{(\ell)}(r)$ in terms of $\phi_i^{(\ell)}(r)$:

$$V(r) \phi_i^{(\ell)}(r) = \sum_{j=0}^{N} V_{ji}^{(\ell)}(m) \phi_j^{(\ell)}(r) , \quad \ell = 0, 1 .$$

Quite obviously, here $V_{ij}^{(\ell)}(m)$ is the real and symmetric matrix of expectation values of the interaction potential $V(r)$ with respect to our basis functions $\phi_i^{(\ell)}(r)$ for a given ℓ:

$$V_{ij}^{(\ell)}(m) = \int_0^\infty dr\, r^2\, V(r)\, \phi_i^{(\ell)}(r)\, \phi_j^{(\ell)}(r) . \tag{2}$$

In Refs. [5, 6, 7] it has been shown that, for interaction potentials of the power-law form

$$V(r) = \sum_n a_n\, r^{b_n}$$

(with sets of arbitrary real constants a_n and b_n), the expectation values $V_{ij}^{(\ell)}(m)$ can be easily worked out algebraically; for their algebraic expression for the most general case, consult either Sec. 4 of Ref. [5], or Sec. 3.10 of Ref. [6], or Sec. 2.8.1 of Ref. [7].

With the aid of the above series expansions, the Fourier–Bessel transformations (6), and the definition (2) of the matrix elements $V_{ij}^{(\ell)}(m)$ of the interaction potential $V(r)$, the matrix $\mathcal{M}_{ij}$ is approximated by the (at least, for all power-law potentials) algebraic expression

$$
\begin{aligned}
\mathcal{M}_{ij} \;=\; & 4\, K_{ij}(m) + 2 \sum_{r=0}^{N} b_{ri}(m)\, V_{rj}^{(0)}(m) + 2\, m^2 \sum_{r=0}^{N} e_{ri}(m)\, V_{rj}^{(0)}(m) \\[4pt]
+ \;& 2 \sum_{r=0}^{N} \sum_{s=0}^{N} c_{ri}^*\, V_{rs}^{(1)}(m)\, d_{sj}(m) + m^2 \sum_{r=0}^{N} \sum_{s=0}^{N} \sum_{t=0}^{N} e_{ri}(m)\, V_{rs}^{(0)}(m)\, e_{st}(m)\, V_{tj}^{(0)}(m) \\[4pt]
+ \;& \sum_{r=0}^{N} \sum_{s=0}^{N} \sum_{t=0}^{N} c_{ri}^*\, V_{sr}^{(1)}(m)\, c_{st}\, V_{tj}^{(0)}(m) .
\end{aligned}
\tag{3}
$$

The explicit evaluation of the matrix elements $K_{ij}(m)$ of the kinetic energy squared and of the various expansion coefficients $b_{ij}(m)$, c_{ij}, $d_{ij}(m)$, and $e_{ij}(m)$ is a tedious but straightforward task. One obtains, for $K_{ij}(m)$ (cf. Sec. 2.2 of Ref. [1]),

$$
\begin{aligned}
K_{ij}(m) \;=\; & \frac{4\, m^2}{\pi\, \sqrt{(i+1)\,(i+2)\,(j+1)\,(j+2)}} \\[6pt]
& \times\; \sum_{r=0}^{i} \sum_{s=0}^{j} (-2)^{r+s} \binom{i+2}{i-r} \binom{j+2}{j-s} (r+1)\,(s+1) \\[6pt]
& \times\; \Bigg[\sum_{k=0}^{|r-s|} \binom{|r-s|}{k} \frac{\Gamma(\tfrac{1}{2}(k+1))\,\Gamma(\tfrac{1}{2}(1+r+s+|r-s|-k))}{\Gamma(\tfrac{1}{2}(2+r+s+|r-s|))} \cos\!\left(\frac{k\,\pi}{2}\right) \\[6pt]
& \qquad - \sum_{k=0}^{r+s+4} \binom{r+s+4}{k} \frac{\Gamma(\tfrac{1}{2}(k+1))\,\Gamma(\tfrac{1}{2}(5+2r+2s-k))}{\Gamma(3+r+s)} \cos\!\left(\frac{k\,\pi}{2}\right) \Bigg] ,
\end{aligned}
$$

or, in matrix form,

$$K(m) \equiv (K_{ij}(m)) = 2\,m^2 \begin{pmatrix} 1 & \dfrac{1}{\sqrt{3}} & \cdots \\[2mm] \dfrac{1}{\sqrt{3}} & \dfrac{5}{3} & \cdots \\[2mm] \vdots & \vdots & \ddots \end{pmatrix},$$

and, for the expansion coefficients $b_{ij}(m)$,

$$
\begin{aligned}
b_{ij}(m) \;=\;& \frac{4\,m}{\pi\,\sqrt{(i+1)\,(i+2)\,(j+1)\,(j+2)}} \\[2mm]
&\times\; \sum_{r=0}^{i} \sum_{s=0}^{j} (-2)^{r+s} \binom{i+2}{i-r} \binom{j+2}{j-s} (r+1)\,(s+1) \\[2mm]
&\times\; \left[\sum_{k=0}^{|r-s|} \binom{|r-s|}{k} \frac{\Gamma(\tfrac{1}{2}\,(k+1))\,\Gamma(\tfrac{1}{2}\,(2+r+s+|r-s|-k))}{\Gamma(\tfrac{1}{2}\,(3+r+s+|r-s|))} \cos\left(\frac{k\,\pi}{2}\right) \right. \\[2mm]
&\left. -\; \sum_{k=0}^{r+s+4} \binom{r+s+4}{k} \frac{\Gamma(\tfrac{1}{2}\,(k+1))\,\Gamma(\tfrac{1}{2}\,(6+2\,r+2\,s-k))}{\Gamma(\tfrac{1}{2}\,(7+2\,r+2\,s))} \cos\left(\frac{k\,\pi}{2}\right) \right]
\end{aligned}
$$

or, in matrix form,

$$b(m) \equiv (b_{ij}(m)) = \frac{64\,m}{5\pi} \begin{pmatrix} \dfrac{1}{3} & \dfrac{1}{7\sqrt{3}} & \cdots \\[2mm] \dfrac{1}{7\sqrt{3}} & \dfrac{11}{27} & \cdots \\[2mm] \vdots & \vdots & \ddots \end{pmatrix}.$$

In order to evaluate the expansion coefficients c_{ij} and $e_{ij}(m)$, we introduce the integrals

$$I_{ij}^{(n)}(m) \;\equiv\; \int_0^{\infty} \frac{\mathrm{d}k\,k^{2+n}}{E(k)}\, \phi_i^{(0)}(k)\,\phi_j^{(0)}(k)\,, \quad n = 0,1,2,\dots\,,$$

$$J_{ij}^{(n)}(m) \;\equiv\; \int_0^{\infty} \frac{\mathrm{d}k\,k^{2+n}}{E(k)}\, \phi_i^{*(1)}(k)\,\phi_j^{(0)}(k)\,, \quad n = 0,1,2,\dots\,; \tag{4}$$

the (somewhat lengthy) explicit expressions of these integrals are given in Appendix B. From the latter, the expansion coefficients c_{ij} and $e_{ij}(m)$ are derived by restricting n to the appropriate values: $c_{ij} = J_{ij}^{(1)}$ and $e_{ij}(m) = I_{ij}^{(0)}(m)$. Explicitly, these matrices read

$$c \equiv (c_{ij}) \;=\; \mathrm{i}\,\frac{1024}{9!!\,\pi} \begin{pmatrix} \sqrt{3} & \dfrac{7}{11} & \cdots \\[2mm] \dfrac{\sqrt{15}}{11} & \dfrac{113\sqrt{5}}{143} & \cdots \\[2mm] \vdots & \vdots & \ddots \end{pmatrix},$$

$$e(m) \equiv (e_{ij}(m)) = \frac{256}{7!!\,\pi\,m} \begin{pmatrix} 1 & -\dfrac{1}{3\sqrt{3}} & \cdots \\ -\dfrac{1}{3\sqrt{3}} & \dfrac{89}{99} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$(2\,n+1)!! \equiv 1 \times 3 \times \cdots \times (2\,n-1) \times (2\,n+1)\,, \quad n = 0, 1, 2, \dots\,.$$

The analytic results for the expansion coefficients $d_{ij}(m)$ have already been derived in Ref. [1]; there is no need to duplicate this formula or the matrix $d(m) \equiv (d_{ij}(m))$ here.

As has been done in Ref. [1], we shall adopt, as the simplest model for a confining interaction between the bound-state constituents, a linear potential: $V(r) = \lambda\,r$, $\lambda > 0$. For this interaction, the general expression for $V_{ij}^{(\ell)}(m)$ given in Refs. [5, 6, 7] simplifies to (cf. Sec. 2.2 of Ref. [1])

$$V_{ij}^{(\ell)}(m, \lambda) = \sqrt{\frac{i!\,j!}{\Gamma(2\,\ell+i+3)\,\Gamma(2\,\ell+j+3)}} \frac{\lambda}{2\,m} \sum_{r=0}^{i} \sum_{s=0}^{j} \frac{(-1)^{r+s}}{r!\,s!}$$

$$\times \begin{pmatrix} i+2\,\ell+2 \\ i-r \end{pmatrix} \begin{pmatrix} j+2\,\ell+2 \\ j-s \end{pmatrix} \Gamma(2\,\ell+r+s+4)\,;$$

the explicit potential matrices $V^{(\ell)}(m, \lambda) \equiv \left(V_{ij}^{(\ell)}(m, \lambda)\right)$, $\ell = 0, 1$, are given in Ref. [1].

The dependence of all these quantities on the dimensional parameters in the theory, viz., the mass m of the bound-state constituents and the slope λ of the linear potential, may be inferred already on dimensional grounds:

$$K_{ij}(m) = m^2\,K_{ij}(1)\,,$$

$$b_{ij}(m) = m\,b_{ij}(1)\,,$$

$$d_{ij}(m) = m\,d_{ij}(1)\,,$$

$$e_{ij}(m) = \frac{1}{m}\,e_{ij}(1)\,,$$

$$V_{ij}^{(\ell)}(m, \lambda) = \frac{\lambda}{m}\,V_{ij}^{(\ell)}(1, 1)\,, \quad \ell = 0, 1, \dots\,;$$

the expansion coefficients c_{ij} are independent of the bound-state constituents' mass m. Factorizing off the dependence on m and λ in the matrix $\mathcal{M}_{ij}$, Eq. (3), we end up with

$$\mathcal{M}_{ij} = 4\,m^2\,K_{ij}(1) + 2\,\lambda \sum_{r=0}^{N} [b_{ri}(1) + e_{ri}(1)]\,V_{rj}^{(0)}(1, 1)$$

$$+ \quad 2\,\lambda \sum_{r=0}^{N} \sum_{s=0}^{N} c_{ri}^*\,V_{rs}^{(1)}(1, 1)\,d_{sj}(1)$$

$$+ \quad \frac{\lambda^2}{m^2} \sum_{r=0}^{N} \sum_{s=0}^{N} \sum_{t=0}^{N} e_{ri}(1)\,V_{rs}^{(0)}(1, 1)\,e_{st}(1)\,V_{tj}^{(0)}(1, 1)$$

$$+ \quad \frac{\lambda^2}{m^2} \sum_{r=0}^{N} \sum_{s=0}^{N} \sum_{t=0}^{N} c_{ri}^*\,V_{sr}^{(1)}(1, 1)\,c_{st}\,V_{tj}^{(0)}(1, 1)\,.$$

Approximate solutions of the Bethe–Salpeter equation are found by diagonalizing $\mathcal{M}_{ij}$.

3 A Few Illustrative Results

Mimicking the analysis of Ref. [1], let us investigate first the case $d = 1$ and $N = 0$. For $i = j = 0$, we find, for the first elements of the matrices $K(1)$, $b(1)$, c, $d(1)$, and $e(1)$,

$$K_{00}(1) = 2 \,, \quad b_{00}(1) = \frac{64}{15\,\pi} \,, \quad c_{00} = \mathrm{i}\,\frac{1024}{315\,\sqrt{3}\,\pi} \,, \quad d_{00}(1) = \mathrm{i}\,\frac{\sqrt{3}}{2} \,, \quad e_{00}(1) = \frac{256}{105\,\pi} \,,$$

and, for the expectation values $V^{(\ell)}(1, 1)$, $\ell = 0, 1$, of the linear potential (cf. Sec. 3 of Ref. [1]),

$$V_{00}^{(0)}(1, 1) = \frac{3}{2} \,, \quad V_{00}^{(1)}(1, 1) = \frac{5}{2} \,.$$

These matrix elements yield, for the bound-state mass M squared, the analytic result

$$M^2 = 8\,m^2 + \frac{8896}{315\,\pi}\,\lambda + \frac{23}{7}\left(\frac{128\,\lambda}{45\,\pi\,m}\right)^2 \qquad (m \neq 0) \,.$$

From this formula, we obtain, for a bound-state constituents' mass $m = 0.9$ GeV and a typical value $\lambda = 0.2$ GeV^2 of the slope of the linear potential [8], for the bound-state mass $M = 2.900$ GeV. This is only 10% away from the "exact" result $M = 2.637$ GeV for the ground-state mass, computed for a matrix size $d = 15$ (that is, a 15×15 matrix) and $N = 49$ (that is, taking into account the first 50 basis functions) in the expansions performed at intermediate steps.

In general, that is, for matrix sizes $d > 4$, the diagonalization of our matrix $\mathcal{M}_{ij}$ can be done only numerically. Table 1 illustrates, for the lowest-lying radial excitations, the rather rapid convergence of the bound-state masses M, obtained as square roots of the eigenvalues of $\mathcal{M}_{ij}$, with increasing matrix size d to the numerically computed "exact" results.

Table 1: Differences $M - 2\,m$ of the eigenvalues M of the instantaneous Bethe–Salpeter equation and the sum of the masses m of the bound-state constituents, in units of GeV, for two spin-$\frac{1}{2}$ fermions of mass $m = 0.1$ GeV, experiencing an interaction described by a linear potential with slope $\lambda = 0.2$ GeV^2 and forming bound states of radial quantum number $n_\mathrm{r} = 0, 1, 2$ and spin-parity-charge conjugation assignment $J^{PC} = 0^{-+}$ (called $1^1\mathrm{S}_0$, $2^1\mathrm{S}_0$, and $3^1\mathrm{S}_0$ in the usual spectroscopic notation) as functions of the matrix size d, for $N = 49$ (i.e., considering 50 basis vectors) in the intermediate series expansions. The last row compares these differences with the outcome of a numerical computation.

d	$1^1\mathrm{S}_0$	$2^1\mathrm{S}_0$	$3^1\mathrm{S}_0$
15	1.477	2.147	2.918
25	1.461	2.095	2.698
50	1.461	2.074	2.560
purely numerical	1.4613	2.0740	2.5580

Figure 1 shows the dependence of the binding energies (i.e., the differences $M - 2\,m$ of the eigenvalues M of the instantaneous Bethe–Salpeter equation and the sum of the masses m of the two bound-state constituents) of the lowest-lying bound states on m. We observe perfect agreement of our findings with results presented in Fig. 1 of Ref. [9].
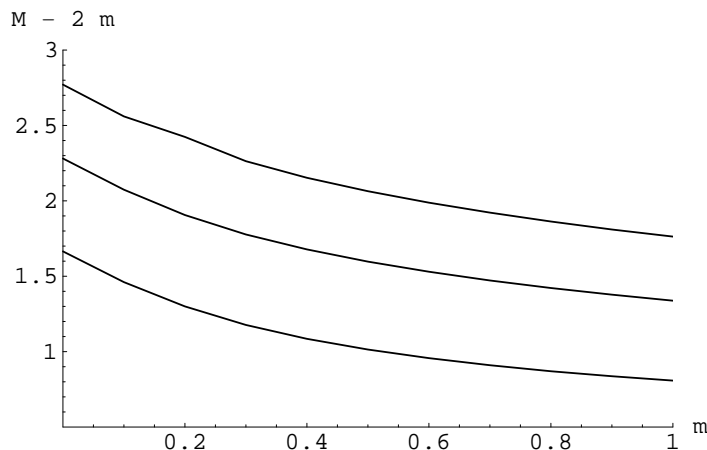
Figure 1: Dependence of the differences $M - 2\,m$ of the eigenvalues M of the instantaneous Bethe–Salpeter equation and the sum of the masses m of the bound-state constituents for the three lowest-lying $J^{PC} = 0^{-+}$ bound states, obtained from a time-component Lorentz-vector confining interaction kernel involving a linear potential $V(r) = \lambda\,r$ with slope $\lambda = 0.2$ GeV^2, on the bound-state constituents' mass m (all masses in units of GeV). The chosen truncation parameters are $N = 49$ in the intermediate series expansions as well as $d = 15$ for the matrix size (except for $m = 0.1$ GeV, where, in order to increase the accuracy, $d = 50$ has been used).

4 Summary, Conclusions, and Outlook

In the present investigation we developed a technique for the approximative solution of the instantaneous Bethe–Salpeter equation with massive bound-state constituents, by a reformulation of this equation of motion as an equivalent matrix eigenvalue problem. Combining these findings with the analoguous result obtained within a similar analysis for the slightly simpler case of massless bound-state constituents presented in Ref. [1], we arrive at the conclusion that, for a suitable choice of basis states in the Hilbert space of solutions, it is for a large class of interactions possible to convert the Bethe–Salpeter equation in the instantaneous approximation for the involved interaction kernel into an eigenvalue problem for an explicitly known matrix, with matrix elements given in form of analytic expressions. The main advantage of our formalism is that, due to the scaling behaviour of the involved quantities, in actual applications (like fitting procedures) the required matrices must be calculated only once (for, e.g., unit values of all the physical parameters). As consequence of the explicit knowledge of the matrix representation of the instantaneous Bethe–Salpeter equation, the eventual diagonalization of this matrix represents the only numerical operation required by this method of solution.

The next step must be to apply this formalism to realistic (i.e., phenomenologically acceptable) models of the interquark forces, capable of reproducing the experimentally observed hadron spectra and features by describing hadrons as bound states of quarks.

Acknowledgement

A   The "Generalized Laguerre" Basis

Our choice of basis states for $L_2(R^+)$ is fixed by the configuration-space representation

$$\phi_i^{(\ell)}(r) = \sqrt{\frac{(2\,m)^{2\,\ell+3}\,i!}{\Gamma(2\,\ell+i+3)}}\,r^\ell \exp(-m\,r)\,L_i^{(2\,\ell+2)}(2\,m\,r)\,, \quad i = 0, 1, 2, \dots\,, \qquad (5)$$

which involves the generalized Laguerre polynomials $L_i^{(\gamma)}(x)$ (for the parameter γ): the latter quantities are orthogonal polynomials which are defined by the power series [4]

$$L_i^{(\gamma)}(x) = \sum_{t=0}^{i} (-1)^t \left(\begin{array}{c} i+\gamma \\ i-t \end{array}\right) \frac{x^t}{t!}\,, \quad i = 0, 1, 2, \dots\,,$$

and which are orthonormalized, with the weight function $x^\gamma \exp(-x)$, according to [4]

$$\int_0^\infty dx\, x^\gamma \exp(-x)\, L_i^{(\gamma)}(x)\, L_j^{(\gamma)}(x) = \frac{\Gamma(\gamma+i+1)}{i!}\, \delta_{ij}\,, \quad i, j = 0, 1, 2, \dots\,.$$

The necessary normalizability of the Hilbert-space basis functions $\phi_i^{(\ell)}(r)$ is guaranteed by the positive numerical value of the mass m of the bound-state constituents: $m > 0$. The basis functions $\phi_i^{(\ell)}(r)$ defined by Eq. (5) satisfy the orthonormalization condition

$$\int_0^\infty dr\, r^2\, \phi_i^{(\ell)}(r)\, \phi_j^{(\ell)}(r) = \delta_{ij}\,, \quad i, j = 0, 1, 2, \dots\,.$$

Note that the configuration-space representation of our basis states is chosen to be real. The corresponding momentum-space representation $\phi_i^{(\ell)}(p)$ of the $L_2(R^+)$ basis states under consideration is obtained from Eq. (5) by a Fourier–Bessel transformation (recall that one is dealing here exclusively with the *radial* parts of the $L_2(R^3)$ basis functions):

$$\phi_i^{(\ell)}(r) = \mathrm{i}^\ell \sqrt{\frac{2}{\pi}} \int_0^\infty dp\, p^2\, j_\ell(p\,r)\, \phi_i^{(\ell)}(p)\,, \quad i = 0, 1, 2, \dots\,, \quad \ell = 0, 1, 2, \dots\,, \qquad (6)$$

$$\phi_i^{(\ell)}(p) = (-\mathrm{i})^\ell \sqrt{\frac{2}{\pi}} \int_0^\infty dr\, r^2\, j_\ell(p\,r)\, \phi_i^{(\ell)}(r)\,, \quad i = 0, 1, 2, \dots\,, \quad \ell = 0, 1, 2, \dots\,.$$

Explicitly, it reads

$$\phi_i^{(\ell)}(p) = \sqrt{\frac{(2\,m)^{2\,\ell+3}\,i!}{\Gamma(2\,\ell+i+3)}}\, \frac{(-\mathrm{i})^\ell\, p^\ell}{2^{\ell+1/2}\,\Gamma\left(\ell+\frac{3}{2}\right)}$$

$$\times\, \sum_{t=0}^{i} \frac{(-1)^t}{t!} \left(\begin{array}{c} i+2\,\ell+2 \\ i-t \end{array}\right) \frac{\Gamma(2\,\ell+t+3)\,(2\,m)^t}{(p^2+m^2)^{(2\,\ell+t+3)/2}}$$

$$\times\, F\left(\frac{2\,\ell+t+3}{2},\, -\frac{1+t}{2};\, \ell+\frac{3}{2};\, \frac{p^2}{p^2+m^2}\right)\,, \quad i = 0, 1, 2, \dots\,, \qquad (7)$$

with the hypergeometric series F, defined, in terms of the gamma function Γ, by [4]

$$F(u, v; w; z) = \frac{\Gamma(w)}{\Gamma(u)\,\Gamma(v)} \sum_{n=0}^{\infty} \frac{\Gamma(u+n)\,\Gamma(v+n)}{\Gamma(w+n)}\, \frac{z^n}{n!}\,.$$

The momentum-space basis functions $\phi_i^{(\ell)}(p)$ satisfy the orthonormalization condition

$$\int\limits_0^\infty \mathrm{d}p\, p^2\, \phi_i^{*(\ell)}(p)\, \phi_j^{(\ell)}(p) = \delta_{ij}\ , \quad i,j = 0,1,2,\dots\ .$$

The availability of the Fourier transform of our basis functions $\phi_i^{(\ell)}(r)$ in analytic form represents the main advantage of our choice (5). Note that the momentum-space basis functions are real for $\ell = 0$, as well as for all even values of ℓ:

$$\phi_i^{*(\ell)}(p) = \phi_i^{(\ell)}(p) \quad \text{for } \ell = 0,2,4,\dots\ , \quad \forall\ i = 0,1,2,\dots\ .$$

In order to get rid of that rather difficult-to-handle hypergeometric series F in Eq. (7), we occasionally take advantage of a somewhat simplified form of the momentum-space basis functions $\phi_i^{(\ell)}(p)$, namely, for $\ell = 0$ [5],

$$\begin{aligned}
\phi_i^{(0)}(p) &= \sqrt{\frac{i!}{m\,\pi\,\Gamma(i+3)}}\,\frac{4}{p}\sum_{t=0}^i (-2)^t\,(t+1)\begin{pmatrix} i+2 \\ i-t \end{pmatrix} \\
&\quad \times \left(1 + \frac{p^2}{m^2}\right)^{-(t+2)/2}\sin\left((t+2)\arctan\frac{p}{m}\right),
\end{aligned}$$

and, for $\ell = 1$,

$$\begin{aligned}
\phi_i^{(1)}(p) &= -\mathrm{i}\sqrt{\frac{m^5}{\pi\,(i+1)\,(i+2)\,(i+3)\,(i+4)}}\,\frac{8}{p^2} \\
&\quad \times \sum_{t=0}^i \frac{(-2)^t}{t!}\begin{pmatrix} i+4 \\ i-t \end{pmatrix}\frac{(t+3)!\,m^t}{(p^2+m^2)^{(t+3)/2}} \\
&\quad \times \left[\frac{\sqrt{p^2+m^2}}{t+2}\sin\left((t+2)\arctan\frac{p}{m}\right) - \frac{m}{t+3}\sin\left((t+3)\arctan\frac{p}{m}\right)\right].
\end{aligned}$$

The latter form of the basis functions is obtained with the help of a suitable recursion formula [4].

B   Some Useful Integrals

With the help of the simplified forms of the basis functions $\phi_i^{(0)}(p)$ and $\phi_i^{(1)}(p)$ recalled in Appendix A, explicit expressions for the integrals defined in Eq. (4) may be found:

$$\begin{aligned}
&I_{ij}^{(n)}(m) \\
&= \frac{4\,m^{n-1}}{\pi\,\sqrt{(i+1)\,(i+2)\,(j+1)\,(j+2)}} \\
&\quad \times \sum_{r=0}^i \sum_{s=0}^j (-2)^{r+s}\begin{pmatrix} i+2 \\ i-r \end{pmatrix}\begin{pmatrix} j+2 \\ j-s \end{pmatrix}(r+1)\,(s+1) \\
&\quad \times \left[\sum_{k=0}^{\lceil |r-s|\rceil}\begin{pmatrix} |r-s| \\ k \end{pmatrix}\frac{\Gamma(\tfrac{1}{2}(k+n+1))\,\Gamma(\tfrac{1}{2}(4+r+s+|r-s|-n-k))}{\Gamma(\tfrac{1}{2}(5+r+s+|r-s|))}\cos\left(\frac{k\pi}{2}\right)\right. \\
&\quad \left. - \sum_{k=0}^{r+s+4}\begin{pmatrix} r+s+4 \\ k \end{pmatrix}\frac{\Gamma(\tfrac{1}{2}(k+n+1))\,\Gamma(\tfrac{1}{2}(8+2r+2s-n-k))}{\Gamma(\tfrac{1}{2}(9+2r+2s))}\cos\left(\frac{k\pi}{2}\right)\right]
\end{aligned}$$

and

$$J_{ij}^{(n)}(m)$$

$$= i\,\frac{8\,m^{n-1}}{\pi\,\sqrt{(i+1)\,(i+2)\,(i+3)\,(i+4)\,(j+1)\,(j+2)}}$$

$$\times \sum_{r=0}^{i}\sum_{s=0}^{j}(-2)^{r+s}\,(r+1)\,(r+2)\,(r+3)\,(s+1)\begin{pmatrix} i+4 \\ i-r \end{pmatrix}\begin{pmatrix} j+2 \\ j-s \end{pmatrix}$$

$$\times\left\{\frac{1}{r+2}\left[\sum_{k=0}^{\lceil |r-s|}\begin{pmatrix} |r-s| \\ k \end{pmatrix}\frac{\Gamma(\frac{1}{2}\,(n+k))\,\Gamma(\frac{1}{2}\,(5+r+s+|r-s|-n-k))}{\Gamma(\frac{1}{2}\,(5+r+s+|r-s|))}\right.\right.$$

$$\times\cos\left(\frac{k\,\pi}{2}\right)$$

$$-\sum_{k=0}^{4+r+s}\begin{pmatrix} 4+r+s \\ k \end{pmatrix}\frac{\Gamma(\frac{1}{2}\,(n+k))\,\Gamma(\frac{1}{2}\,(9+2\,r+2\,s-n-k))}{\Gamma(\frac{1}{2}\,(9+2\,r+2\,s))}\cos\left(\frac{k\,\pi}{2}\right)\Bigg]$$

$$-\frac{1}{r+3}\left[\sum_{k=0}^{\lceil |1+r-s|}\begin{pmatrix} |1+r-s| \\ k \end{pmatrix}\right.$$

$$\times\frac{\Gamma(\frac{1}{2}\,(n+k))\,\Gamma(\frac{1}{2}\,(6+r+s+|1+r-s|-n-k))}{\Gamma(\frac{1}{2}\,(6+r+s+|1+r-s|))}\cos\left(\frac{k\,\pi}{2}\right)$$

$$-\sum_{k=0}^{5+r+s}\begin{pmatrix} 5+r+s \\ k \end{pmatrix}\frac{\Gamma(\frac{1}{2}\,(n+k))\,\Gamma(\frac{1}{2}\,(11+2\,r+2\,s-n-k))}{\Gamma(\frac{1}{2}\,(11+2\,r+2\,s))}\cos\left(\frac{k\,\pi}{2}\right)\Bigg]\Bigg]\Bigg\}.$$

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